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ON CERTAIN METHODS OF SOLVING SYSTEMS OF INTEGRODIFFERENTIAL EQUATIONS ENCOUNTERED IN VISCOELASTICITY PROBLEMS*

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The method of freezing proposed and given a foundation by A.N. Filatov, for systems of integrodifferential equations (IDE) of standard form /1-4/ is applied to IDE systems encountered in dynamic viscoelasticity problems. A numerical method is proposed for IDE systems, which is based on using quadrature formulas. A specific example is examined to compare this method with other known methods (the method of averaging and the method of freezing). Furthermore, a problem on the longitudinal vibrations of a viscoelastic rod in a physically non-linear formulation is investigated by the method of freezing in combination with a numerical Runge-Kutta method.

1. Let us consider an IDE system of the form

$$T_i^{\cdot\cdot} + \omega_i^2 T_i = f_i(t) + \mu X_i\left(t, T_1, \ldots, T_n, \int_0^t \varphi_i(t, \tau, T_1(\tau), \ldots, T_n(\tau))\, d\tau\right) \tag{1.1}$$
$$T_i(0) = T_{0i}, \qquad T_i^{\cdot}(0) = T_{0i}^{\cdot}$$

Here $T_i(t)$ is the desired function of the argument t, $\mu > 0$ is a small parameter, f_i, X_i and φ_i are given continuous functions in the range of variation of the arguments, and the subscript i takes on the values $1, 2, \ldots, n$ everywhere.

By making the substitution

$$T_i(t) = C_{1i} \cos \omega_i t + C_{2i} \sin \omega_i t + \frac{1}{\omega_i} \int_0^t f_i(\tau) \sin \omega_i (t-\tau)\, d\tau \tag{1.2}$$

we can reduce system (1.1) to standard form. Applying the freezing procedure /1-4/ to the system obtained and taking account of relationship (1.2), we obtain after differentiation

$$T_i^{\cdot\cdot} + \omega_i^2 T_i = f_i(t) + \mu X_i\Big\{t, T_1, \ldots, T_n, \int_0^t \varphi_i\Big(t, t-\tau, T_1(t) \cos \omega_1 \tau - \tag{1.3}$$
$$\frac{1}{\omega_1} T_1^{\cdot}(t) \sin \omega_1 \tau - \frac{1}{\omega_1} \int_{t-\tau}^t f_1(s) \sin \omega_1 (t-\tau-s)\, ds, \ldots, T_n(t) \cos \omega_n \tau -$$
$$\frac{1}{\omega_n} T_n^{\cdot}(t) \sin \omega_n \tau - \frac{1}{\omega_n} \int_{t-\tau}^t f_n(s) \sin \omega_n (t-\tau-s)\, ds\Big) d\tau\Big\}$$
$$T_i(0) = T_{0i}, \qquad T_i^{\cdot}(0) = T_{0i}^{\cdot}$$

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Therefore, a system of differential equations with variable coefficients of the form (1.3) is set in correspondence with system (1.1) by the method of freezing. It is obviously simpler to investigate system (1.3) than the IDE system of the form (1.1). Moreover, well-known numerical integration methods can be applied to system (1.3).

The form of system (1.3) enables one to perform the freezing action without reducing the initial IDE system to standard form.

2. We now apply a numerical method based on utilization of quadrature formulas for system (1.1). We will write this system in integral form. Then setting $t = t_m, t_j = jh, j = 1, \ldots, m; m = 1, 2, \ldots (h = \text{const})$, and replacing the integrals by certain quadrature formulas, we obtain the following approximate formula to calculate the values of $T_{mi} = T_i(t_m)$:

$$T_{mi} = T_{0i}\cos\omega_i t_m + \frac{1}{\omega_i}T_{0i}^{\cdot}\sin\omega_i t_m + \frac{1}{\omega_i}\sum_{j=0}^{m-1}A_j f_i(t_j)\sin\omega_i(t_m - t_j) + \tag{2.1}$$
$$\frac{\mu}{\omega_i}\sum_{j=0}^{m-1}B_j X_i\Big(t_j, T_{j1}, \ldots, T_{jn}, \sum_{k=0}^{j}C_k\varphi_i(t_j, t_k, T_{k1}, \ldots, T_{kn})\Big)\times$$
$$\sin\omega_i(t_m - t_j), \quad m = 1, 2, \ldots$$

where A_j, B_j and $C_k, j = 0, 1, \ldots, m; k = 0, 1, \ldots, j; m = 1, 2, \ldots$ are numerical coefficients independent of the integrand selections and taking on different values depending on the quadrature formulas utilized.

The form of the integrands enables one to find numerical values of the desired functions sequentially from (2.1) by using the given initial conditions. The error in the method proposed agrees with the error obtained when using quadrature formulas and is of the same order of smallness relative to the interpolation step.

3. To compare the method of freezing with certain other methods, we consider the Cauchy problem

$$T^{\cdot\cdot} + \omega^2 T = f(t) + \omega^2\int_0^t R(t-\tau)T(\tau)\,d\tau \tag{3.1}$$
$$T(0) = T_0, \quad T^{\cdot}(0) = T_0^{\cdot}$$

We will show that many viscoelasticity problems reduce to equations of the form (3.1). As is well-known /5/, the dynamical problems of linear viscoelasticity theory reduce, after application of Bubnov-Galerkin type methods in the space variables (finite elements), to IDE systems of the form

$$\mathbf{M}u^{\cdot\cdot} + \mathbf{G}u = \mathbf{F}(t) + \mathbf{G}\int_0^t R(t-\tau)u(\tau)\,d\tau \tag{3.2}$$

(**M** is the matrix of the system mass, **G** is the stiffness matrix, and $R(t)$ is the relaxation kernel). If the matrix of the nodal displacements $u(t)$ is represented in the form of linear combinations of the eigenvectors of the corresponding elastic problems, i.e., /6/

$$u(t) = T_1(t)W_1 + T_2(t)W_2 + \ldots$$
$$W_i^T\mathbf{M}W_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j), \end{cases} \quad W_j^T\mathbf{G}W_j = \begin{cases} 0 & (i \neq j) \\ \omega_i^2 & (i = j) \end{cases}$$

then by omitting the subscripts we obtain (3.1) from (3.2).

According to (1.3), the appropriate differential equation for (3.1) can be written in the form

$$T^{\cdot\cdot} + \omega\Gamma_s(t)T^{\cdot} + \omega^2[1 - \Gamma_c(t)]T = f(t) -$$
$$\omega\int_0^t R(\tau)\,d\tau\int_{t-\tau}^t f(s)\sin\omega(t-\tau-s)\,ds$$
$$T(0) = T_0, \quad T(0) = T_0^{\cdot}$$
$$\left(\Gamma_s(t) = \int_0^t R(s)\sin\omega s\,ds,\ \Gamma_c(t) = \int_0^t R(s)\cos\omega s\,ds\right)$$

The differential equation with constant coefficients

$$T^{\cdot\cdot} + {}^1/_2\omega\Gamma_s(\infty)T^{\cdot} + \omega^2[1 - \Gamma_c(\infty)]T = f(t) + \omega(D_s\sin\omega t + D_c\cos\omega t) -$$
$$\frac{\omega}{2}\int_0^t f(\tau)[\sin\omega(t-\tau) - \cos\omega(t-\tau)]\,d\tau$$

$$T(0) = T_0, \quad T^{\cdot}(0) = T_0^{\cdot}$$

$$\begin{Bmatrix} D_s \\ D_c \end{Bmatrix} = \lim_{p\to\infty} \frac{1}{p} \int_0^p D(t) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \omega t\, dt$$

$$D(t) = \int_0^\infty R(\tau)\, d\tau \int_0^{t-\tau} \sin\omega(t-\tau-s)\, ds$$

is set in correspondence with (3.1) by the method of averaging /1-4/.

We now apply the method proposed in Sect.2 for (3.1). We write the equation in integral form and (2.1) becomes

$$T_m = T_0 \cos\omega t_m + \frac{1}{\omega} T_0^{\cdot} \sin\omega t_m +$$
$$\frac{1}{\omega}\sum_{j=0}^{m-1} A_j f(t_j) \sin\omega(t_m - t_j) + \omega \sum_{j=0}^{m-1} B_j F(t_m, t_j) T_j, \quad m = 1, 2, \ldots$$
$$F(t, \tau) = \int_0^{t-\tau} R(s) \sin\omega(t-\tau-s)\, ds, \qquad F(t, t) = 0$$

where A_j, B_j $(j = 0, 1, \ldots, m-1)$ are coefficients of the Simpson quadrature formula.

Let us consider (3.1) for such data

$$f(t) = a\sin\theta t, \quad R(t-\tau) = \alpha\exp[-2\beta(t-\tau)]$$
$$\alpha = 2\beta = 0{,}05\ \mathrm{sec}^{-1}, \quad a = 240\ \mathrm{kg/cm}^2, \quad \theta = 2\omega$$
$$\omega^2 = 350\ \mathrm{sec}^{-2}, \quad T_0 = 0{,}05\ \mathrm{cm}, \quad T_0^{\cdot} = 0$$

The following

$$T(t) = \exp(-\beta t)\,[(T_0 - \beta q)\cos\lambda t + (\beta\lambda^{-1}T_0 + e)\sin\lambda t] +$$
$$r\sin\theta t + \beta q\cos\theta t, \quad q = {}^1/_2 a\lambda^{-1}(q_+ - q_-)$$
$$e = {}^1/_2 a\lambda^{-1}(e_+ - e_-), \quad r = {}^1/_2 a\lambda^{-1}(e_+ + e_-)$$
$$e_\pm = (\lambda \pm \theta)\, q_\pm, \quad q_\pm = [(\lambda\pm\theta)^2 + \beta^2]^{-1}, \quad \lambda = \sqrt{\omega^2 - \beta^2}$$

is the exact solution of (3.1) for the $f(t)$, $R(t)$ and initial data taken.

Below we give the exact solution (T) of (3.1) and the approximate solutions obtained by the method of averaging (T_a), the method of freezing (T_b) and the method based on using the quadrature formulas proposed in Sect.2 (T_c)

t	0	2	4	6	8
$T\cdot 10^4$	500	380	44,8	−531	−1344
$T_a\cdot 10^4$	500	424	181,6	−273	−961
$T_b\cdot 10^4$	500	377	38,9	−539	−1353
$T_c\cdot 10^4$	500	380	43,3	−533	−1345

It is seen that for the problem under consideration the method of freezing and the method based on using the quadrature formulas yield more accurate results as compared with the method of averaging.

4. We consider the problem of longitudinal vibrations of a viscoelastic rod. We take the relationship between the stress σ and the strain ε in the form

$$\sigma = E\left\{\varepsilon - \gamma\varepsilon^3 - \int_0^t R(t-\tau)\,[\varepsilon(\tau) - \gamma\varepsilon^3(\tau)]\, d\tau\right\}$$

where $R(t)$ is the relaxation kernel, γ is a non-linearity factor dependent on the rod material, and E is the elastic modulus. Substituting this expression into the vibrations equation, introducing the dimensionless parameters ($u(x, t)$ is the displacement, and ρ is the density of the rod material)

$$u/l, \quad x/l, \quad t(\rho l^2/E)^{-1/2}, \quad R(t)\cdot(\rho l^2/E)^{1/2}$$

and retaining here all the previous notation, we obtain (the prime denotes the derivative with respect to x)

$$u^{\cdot\cdot} = u'' - \int_0^t R(t-\tau)\, u''(x,\tau)\, d\tau - 3\gamma(u')^2 u'' + \tag{4.1}$$
$$3\gamma\int_0^t R(t-\tau)\,(u'(x,\tau))^2\, u''(x,\tau)\, d\tau$$

We supplement (4.1) by the initial conditions

$$u(x, 0) = f_0(x), \; u^{\cdot}(x, 0) = g_0(x)$$

Assuming one end of the rod to be clamped and the other to be free, we seek the solution of (4.1) that will satisfy the boundary conditions of the problem, in the form

$$u(x, t) = \sum_{k=1,3,\ldots}^{2N-1} T_k(t) \sin \frac{k\pi x}{2} \tag{4.2}$$

Substituting (4.2) into (4.1) and applying the Bubnov-Galerkin procedure, we have the following IDE system for finding the desired functions $T_k(t)$ $(k = 1, 3, \ldots, 2N-1)$

$$T_k^{\cdot\cdot} + \omega_k^2 T_k = \omega_k^2 \int_0^t R(t-\tau) T_k(\tau)\, d\tau + \gamma\varphi_k(T_1, T_3, \ldots, T_{2N-1}) - \tag{4.3}$$

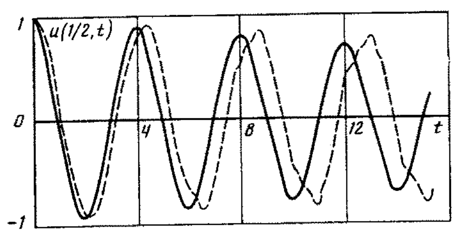
$$\gamma \int_0^t R(t-\tau)\, \varphi_k(T_1(\tau), T_3(\tau), \ldots, T_{2N-1}(\tau))\, d\tau$$

$$\omega_k = {}^1/_2 k\pi, \; \varphi_k(T_1, T_3, \ldots, T_{2N-1}) = \frac{3\pi^4}{8} \int_0^1 \Big[\sum_{m=1,3,\ldots}^{2N-1} mT_m(t) \cos \frac{m\pi x}{2} \Big]^2 \times$$

$$\Big[\sum_{m=1,3,\ldots}^{2N-1} m^2 T_m(t) \sin \frac{m\pi x}{2} \Big] \sin \frac{k\pi x}{2}\, dx$$

The following data are used for the numerical computations:

$$R(t-\tau) = \alpha \exp[-2\beta(t-\tau)], \; \alpha = 2\beta = 0{,}05$$
$$f_0(x) = {}^8/_3 (x - {}^1/_2 x^2), \; g_0(x) = 0$$



According to (1.3), a system of differential equations for whose solution the Runge-Kutta method is used, is set in correspondence to system (4.3) by the method of freezing.

Five of the first harmonics are kept in (4.2) (calculations showed that a further increase in the number of terms exerts substantially no influence on the amplitude of rod vibrations). The vibrations mode of the middle point of the rod is represented in the figure for the values $\gamma = 0$ (the solid line) and $\gamma = 0{,}05$ (the dashes).

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